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Best Approximation and Saturation on Domains Bounded by Curves of Bounded Rotation

I. Bruj

Belarussische Commerzuniversität f. Verwaltung, Pramyslowajastr. 3, Baranawitschy 16, 225320 Brester Gebiet, Belarus

and

G. Schmieder

Fachbereich Mathematik, Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, BR, Germany
E-mail: schmieder@mathematik.uni-oldenburg.de

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Let G be a Jordan domain with a boundary curve of bounded rotation; We consider approximation of complex-valued functions on G and ask for best approximation by certain matrix-means of Faber polynomials and determine the order of saturation concerning this approximation type. © 1999 Academic Press

1. INTRODUCTION

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ stands for the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ for the class of all holomorphic functions $\mathbb{D} \to \mathbb{C}$. By $\bar{\mathbb{D}}$ we denote the closed unit disk and define $C(\bar{\mathbb{D}})$ as the set of functions continuous on $\bar{\mathbb{D}}$ as well as $A(\bar{\mathbb{D}}) := C(\bar{\mathbb{D}}) \cap H(\mathbb{D})$. The rth derivative of a function $f \in H(\mathbb{D})$ we write as $D^r f$. For $\alpha \in\,]0, 1[$ we define

$$D^r H^\alpha(\bar{\mathbb{D}}) := \{ f \in A(\bar{\mathbb{D}}) : (D^r f)(z_1) - (D^r f)(z) \leqslant \text{const} \cdot |z_1 - z_2|^\alpha \text{ for all } z_1, z_2 \in \mathbb{D} \}.$$

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By way of introduction we consider a non constant function $f^* \in D^5 H^\alpha(\bar{\mathbb{D}})$. In 1911, D. Jackson proved

$$E_n(f^*, \bar{\mathbb{D}}) := \inf_{p \in \Pi_n} \max_{z \in \bar{\mathbb{D}}} |f^*(z) - p(z)| = O\left(\frac{1}{n^{5+\alpha}}\right) \qquad (n \to \infty)$$

(Π_n stands for the set of complex polynomials of degree not larger than n). This means $n^{-5-\alpha}$ is the quality of approximation of f^* on $\bar{\mathbb{D}}$ by polynomials of degree less or equal n. Our function f^* has a Taylor expansion $\sum_{m=0}^{\infty} a_m(f^*, \bar{\mathbb{D}})\, z^m$ which is uniformly convergent on $\bar{\mathbb{D}}$. The Rogosinski means are defined as

$$U_n^{\mathrm{Rog}}(f^*, \bar{\mathbb{D}}, z) := \sum_{m=0}^{n} a_m(f^*, \bar{\mathbb{D}}) \left(\cos \frac{m\pi}{2(n+1)}\right) z^m.$$

A result from 1951, independently given by S. B. Stechkin and A. F. Timan, states (cf. [9, 5.11.7 (8)])

$$\max_{z \in \bar{\mathbb{D}}} |f^*(z) - U_n^{\mathrm{Rog}}(f^*, \bar{\mathbb{D}}, z)| = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

This is best possible as we see from a general theorem of M. Zamansky. He proved that if $f \in A(\bar{\mathbb{D}})$ and

$$\max_{n \in \bar{\mathbb{D}}} |f(z) - U_n^{\mathrm{Rog}}(f, \bar{\mathbb{D}}, z)| = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

then $f \equiv \mathrm{const}$. Thus, the order of saturation with resp. to the Rogosinski means is n^{-2}. We go back to the special case of f^*. Information about the quality of approximation between the order of saturation and the order of best approximation, we find in a result of A. K. Pokalo (1957) [6, pp. 751, 753]

$$\begin{aligned} f^*(z) &- U_n^{\mathrm{Rog}}(f^*, \bar{\mathbb{D}}, z) \\ &= \frac{\pi^2}{8} \frac{z(D^1 f^*)(z) + z^2(D^2 f^*)(z)}{(n+1)^2} \\ &\quad - \frac{\pi^4}{384} \frac{z(D^1 f^*)(z) + 7z^2(D^2 f^*)(z) + 6z^3(D^3 f^*)(z) + z^4(D^4 f^*)(z)}{(n+1)^4} \\ &\quad + O\left(\frac{1}{n^{5+\alpha}}\right) \qquad (n \to \infty, z \in \mathbb{D}). \end{aligned} \tag{1.1}$$

The family of Faber series is the natural generalization of the family of Taylor series when the unit disk is replaced by an arbitrary simply connected domain, bounded by a "nice" curve. In this paper we will treat the above question for such a generalized situation. Our main result (Section 6) gives several corollaries including the results cited above.

2. BASIC DEFINITIONS AND RESULTS

We consider a Jordan domain G with rectifiable boundary curve. By $A(\bar{G})$ we denote the class of functions that are holomorphic in G and continuous on the closure $\bar{G}$. Let Π_n be the set of all complex polynomials of degree $0 \leqslant k \leqslant n$. The best polynomial approximation of $f \in A(\bar{G})$ is associated with

$$E_n(f, \bar{G}) := \inf_{p \in \Pi_n} \max_{z \in \bar{G}} |f(z) - p(z)|.$$

Walsh [10, p. 431] proved $E_n(f, \bar{G}) = o(1)$ for every $f \in A(\bar{G})$.

Now let $z = \psi(w)$ be the conformal mapping of $\mathbb{C} \backslash \bar{\mathbb{D}}$ onto $\mathbb{C} \backslash \bar{G}$ normalized by

$$0 < \lim_{w \to \infty} \frac{\psi(w)}{w} < +\infty.$$

The inverse function we denote by $w = \psi^{-1}(z)$. The boundary curve γ of G has a tangent almost everywhere because γ has length $L < +\infty$. Then γ is called of *bounded rotation* if the angle of γ' can be extended to a function of bounded variation on the whole curve (cf. [7, p. 63]).

Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ also viewed as $\mathbb{T} = \partial\mathbb{D} \subset \mathbb{C}$. For $0 < \alpha \leqslant 1$ we define the class $H^\alpha(\mathbb{T})$ of all functions $h\colon \mathbb{T} \to \mathbb{C}$ that fulfill

$$|h(x_1) - h(x_2)| \leqslant \text{const} \cdot |x_1 - x_2|^\alpha \qquad (x_1, x_2 \in \mathbb{T}).$$

Using Pommerenke's results Suetin proved [8, p. 229]: *Let G be a Jordan domain with rectifiable boundary curve of bounded rotation. For $f \in A(\bar{G})$ let $D^r f$ denote the rth derivative of f. Assume that*

$$D^r f \circ \psi \mid \{|w| = 1\} \in H^\alpha(\mathbb{T})$$

for some $0 < \alpha \leqslant 1$. Then $E_n(f, \bar{G}) = O(1/n^{r+\alpha})$ for $n \to \infty$. For $m \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$ we have a Laurent series

$$(\psi^{-1}(\bar{G}, z))^m = a_m^{(m)} z^m + \cdots + a_0^{(m)} + \sum_{k=1}^{\infty} \frac{a_{-k}^{(m)}}{z^k}.$$

The polynomial part of this series is called the Faber polynomial $p_m(\bar{G}, z)$ of $\bar{G}$ (note that p_m has degree m). For $f \in A(\bar{G})$ we define the Faber coefficients of f as

$$a_m(f, \bar{G}) := \frac{1}{2\pi i} \int_{|\tau| = 1} \frac{f(\psi(\bar{G}, \tau))}{\tau^{m+1}} \, d\tau$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi(\bar{G}, e^{it}))\, e^{-imt} \, dt \qquad (m \in \mathbb{N}_0) \qquad (2.0.1)$$

as well as the partial sums

$$s_n(f, \bar{G}, z) := \sum_{m=0}^{n} a_m(f, \bar{G})\, p_m(\bar{G}, z) \qquad (n \in \mathbb{N}_0) \qquad (2.0.2)$$

of the Faber series of f on $\bar{G}$.

THEOREM 1 (Kövari and Pommerenke [5, p. 199; 8, p. 235]). *Let G be a Jordan domain with rectifiable boundary curve of bounded rotation and $f \in A(\bar{G})$. Then*

$$\max_{z \in \bar{G}} |f(z) - s_n(f, \bar{G}, z)| \leqslant (k_1 \ln(n+2) + k_2) \cdot E_n(f, \bar{G}),$$

where k_1, k_2 are constants depending only on $\bar{G}$.

Concerning the Fejér means

$$\sigma_n(f, \bar{G}, z) = \frac{1}{n+1} \sum_{m=0}^{n} s_m(f, \bar{G}, z)$$

$$= \sum_{m=0}^{n} a_m(f, \bar{G}) \left(1 - \frac{m}{n+1}\right) p_m(\bar{G}, z) \qquad (n \in \mathbb{N}_0)$$

we have the following

THEOREM 2 (Gaier [4, p. 54]). *Let G, f as in the preceding theorem and moreover $f \circ \psi \mid \{|w| = 1\} \in H^\alpha$ with some $0 < \alpha < 1$. Then*

$$E_n(f, \bar{G}) \leqslant \max_{z \in \bar{G}} |f(z) - \sigma_n(f, \bar{G}, z)| = O\left(\frac{1}{n^\alpha}\right) \qquad (n \to \infty).$$

In the case $\alpha = 1$ we have, for technical reasons, to consider the Riesz means

$$R_n^2(f, \bar{G}, z) := \sum_{m=0}^{n} a_m(f, \bar{G}) \left(1 - \left(\frac{m}{n+1}\right)^2\right) p_m(\bar{G}, z) \qquad (n \in \mathbb{N}_0) \qquad (2.1)$$

and here we have the result

THEOREM 3 (Gaier [4, p. 54]). *Let G, f as in the preceding theorem and moreover $f \circ \psi \mid \{|w|=1\} \in H^\alpha$ with some $0<\alpha\leqslant 1$. Then*

$$E_n(f, \bar{G}) \leqslant \max_{z\in\bar{G}} |f(z)-R_n^2(f, \bar{G}, z)| = O\left(\frac{1}{n^\alpha}\right) \qquad (n\to\infty).$$

3. ORDER OF SATURATION

We consider the infinite matrix

$$\mu=(\mu_m^n)=\begin{pmatrix} \mu_0^0 & \mu_1^0 & \cdots & \mu_m^0 & \cdots \\ \mu_0^1 & \mu_1^1 & \cdots & \mu_m^1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_0^n & \mu_1^n & \cdots & \mu_m^n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and the μ-means

$$U_n^\mu(f, \bar{G}, z) := \sum_{v=0}^{n} a_v(g, \bar{G})\, \mu_v^n\, p_v(\bar{G}, z) \qquad (n\in\mathbb{N}_0).$$

THEOREM 4. *Let $G\subset\mathbb{C}$ a Jordan domain with rectifiable boundary and $f\in A(\bar{G})$ a non-constant function. A matrix μ as above is given with $\mu_m^n\neq 1$ for all $n\in\mathbb{N}$, $m=1, ..., n$. Then*

$$\max_{z\in\bar{G}} |f(z)-U_n^\mu(f, \bar{G}, z)| \neq o(\min_{1\leqslant v\leqslant n} |1-\mu_v^n|) \qquad \textit{as} \quad n\to\infty.$$

Before giving the proof we will discuss the assertion of the theorem by two examples:

EXAMPLE A. First we consider the Fejér means.

We take $\mu_m^n := \max\{0, 1-m/(n+1)\}$ for $n, m\in\mathbb{N}_0$. Here we have $\min_{1\leqslant m\leqslant n} |1-\mu_m^n| = 1/(n+1)$. Thus the Fejér-means approximate the function f on $\bar{G}$ not better than $\min_{1\leqslant m\leqslant n} |1-\mu_m^n| = 1/(n+1)$.

EXAMPLE B. Next we consider the M. Riesz means.

These appear for $\mu_m^n := \max\{0, 1-(m/(n+1))^q\}$ where $n, m\in\mathbb{N}_0$ and q is some fixed number in $]0, \infty[$. In this case the theorem gives $\min_{1\leqslant m\leqslant n} |1-\mu_m^n| = (1/(n+1))^q$ with similar consequence as above.

This leads to the following

DEFINITION 1. Let G, μ as in Theorem 4 and $\varepsilon = \varepsilon(n)\colon \mathbb{N} \to]0, \infty[$ some function which tends monotonically to 0 for $n \to \infty$. Then $\varepsilon(n)$ is called *order of saturation* for the μ-means $U_n^\mu(f, \bar{G}, z)$ of the Faber-expansion of $f \in A(\bar{G})$ if

(1) $\max_{z \in \bar{G}} |f(z) - U_n^\mu(f, \bar{G}, z)| = o(\varepsilon(n))$ implies $f = \mathit{const}$ on $\bar{G}$, and

(2) there exists some non-constant $g \in A(\bar{G})$ such that

$$\max_{z \in \bar{G}} |g(z) - U_n^\mu(g, \bar{G}, z)| = O(\varepsilon(n))$$

for $n \to \infty$ is fulfilled.

Remarks. (1) For the Fejér means (Example A), $\varepsilon(n) = 1/n$ gives the order of saturation.

(2) For the M. Riesz means (Example B), $\varepsilon(n) = 1/n^q$ gives the order of saturation.

(3) For the Rogosinski means, $\varepsilon(n) = 1/n^2$ gives the order of saturation because $(n \to \infty)$

$$1 - \mu_m^n = 1 - \cos \frac{m\pi}{2(n+1)} \sim \frac{m^2\pi^2}{8} \frac{1}{n^2}.$$

Proof of Theorem 4. Since, by [5, p. 198]

$$\frac{1}{2\pi i} \int_{|\tau|=1} \frac{p_n(\bar{G}, \psi(\bar{G}, \tau))}{\tau^{m+1}} \, d\tau = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \qquad (m, n \in \mathbb{N}_0),$$

it follows from the definition of Faber coefficients and the μ-means,

$$(1 - \mu_m^n)\, a_m(f, \bar{G}) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{f(\psi(\bar{G}, \tau)) - U_n^\mu(f, \bar{G}, \psi(\bar{G}, \tau))}{\tau^{m+1}} \, d\tau$$

$$(n \geqslant m \in \mathbb{N}_0)$$

and therefore

$$|1 - \mu_m^n| \, |a_m(f, \bar{G})| \leqslant \max_{z \in \bar{G}} |f(z) - U_n^\mu(f, \bar{G}, z)|.$$

We assume the conclusion of Theorem 4 to be false. Thus

$$\max_{z \in \bar{G}} |f(z) - U_n^\mu(f, \bar{G}, z)| \overset{n \to \infty}{=} o(\min_{1 \leqslant \nu \leqslant n} |1 - \mu_\nu^n|)$$

and we obtain

$$|1-\mu_m^n|\ |a_m(f, \bar{G})| \overset{n \to \infty}{=} o(\min_{1 \leqslant \nu \leqslant n} |1-\mu_\nu^n|)$$

and therefore

$$\frac{|1-\mu_m^n|\ |a_m(f, \bar{G})|}{\min_{1 \leqslant \nu \leqslant n} |1-\mu_\nu^n|} \overset{n \to \infty}{=} o(1)$$

which gives

$$a_m(f, \bar{G}) = 0 \qquad (m \geqslant 1)$$

and a theorem of Gaier [4, p. 44] now shows $f(z) = a_0(f, \bar{G})$ for all $z \in \bar{G}$. This contradicts the assertion.

4. FABER DERIVATIVE IN THE UNIT DISK

By $C(\mathbb{T})$ we denote the class of 2π-periodic continuous functions $g: \mathbb{R} \to \mathbb{C}$ and denote for such functions and $0 < \delta < \infty$, the modulus of continuity by $\omega(g, \mathbb{T}, \delta)$.

Now we consider some $f \in A(\bar{\mathbb{D}})$. Let $g(x) := f(\exp(ix))$. Then $g \in C(\mathbb{T})$, and

$$\int_0^\pi \frac{\omega(g, \mathbb{T}, \delta)}{\delta}\, d\delta < \infty$$

implies [11, Theorem 6.8]

$$g(x) = \sum_{m=0}^{\infty} a_m(f, \bar{\mathbb{D}})\, e^{imx} \qquad (x \in \mathbb{R}).$$

Similarly we get for the rth-order derivative D^r,

$$\int_0^\pi \frac{\omega(D^r g, \mathbb{T}, \delta)}{\delta}\, d\delta < \infty$$

$$\Rightarrow D^r g(x) = \sum_{m=1}^{\infty} (im)^r\, a_m(f, \bar{\mathbb{D}})\, e^{imx} \qquad (x \in \mathbb{R},\ r \geqslant 1)$$

and in both cases the trigonometric series converges uniformly. This observation leads to the

DEFINITION 2. For $f \in A(\overline{\mathbb{D}})$ and $r \in \mathbb{N}$ we define the Faber derivative of rth order as

$$(F^r f)(z) := \sum_{m=1}^{\infty} (im)^r \, a_m(f, \overline{\mathbb{D}}) \, z^m \qquad (z \in \mathbb{D}).$$

Moreover let $F^0 f = D^0 f = f$.

The proof of the following equations is left to the reader:

$$(D^1 f)(z) = \frac{1}{iz} (F^1 f)(z)$$

$$(D^2 f)(z) = \frac{1}{(iz)^2} (F^2 f)(z) - \frac{1}{iz} \frac{1}{z} (F^1 f)(z),$$

$$(D^3 f)(z) = \frac{1}{(iz)^3} (F^3 f)(z) - \frac{3}{(iz^2)} \frac{1}{z} (F^2 f)(z) + \frac{2}{iz} \frac{1}{z^2} (F^1 f)(z),$$

$$\cdots$$

$$\begin{aligned}(D^r f)(z) = {} & \frac{1}{(iz)^r} (F^r f)(z) - \frac{1+2+\cdots+(r-1)}{(iz)^{r-1}} \frac{1}{z} (F^{r-1} f)(z) \\ & + \frac{1\cdot 2 + 1\cdot 3 + \cdots + (r-2)\cdot(r-1)}{(iz)^{r-2}} \frac{1}{z^2} (F^{r-2} f)(z) - \cdots \\ & + (-1)^{r-1} \frac{1\cdot 2 \cdot \cdots \cdot (r-1)}{iz} \frac{1}{z^{r-1}} (F^1 f)(z).\end{aligned} \tag{4.1}$$

So, for $f \in H(\mathbb{D})$, $z \in \mathbb{D}$ we obtain

$$(F^1 f)(z) = i((zD)\, f)(z) := iz(Df)(z)$$

$$(F^2 f)(z) = i^2(z(Df)(z) + z^2(D^2 f)(z))$$

$$(F^3 f)(z) = i^3(z(Df)(z) + 3z^2(D^2 f)(z) + z^3(D^3 f)(z))$$

$$(F^4 f)(z) = i^4(z(Df)(z) + 7z^2(D^2 f)(z) + 6z^3(D^3 f)(z) + z^4(D^4 f)(z))$$

$$\cdots$$

$$(F^r f)(z) = i^r((zD)^r\, f)(z) = i^r(zD(zD)^{r-1}\, f)(z) := i^r(zD(zD)^{r-1}\, f)(z).$$

5. FABER DERIVATIVE IN A JORDAN DOMAIN

Let $f \in A(\bar{\mathbb{D}})$, $r \in \mathbb{N} \cup \{0\}$ and $\varphi := F^r f$. It follows from Definition 2 that $a_m(f, \bar{\mathbb{D}}) = a_m(\varphi, \bar{\mathbb{D}})/(im)^r$ for all $m \in \mathbb{N}$ and so we have

$$f(z) = a_0(f, \bar{\mathbb{D}}) + \sum_{m=1}^{\infty} \frac{a_m(\varphi, \bar{\mathbb{D}})}{(im)^r} z^m \qquad (z \in \mathbb{D})$$

and the power series converges uniformly on $\mathbb{D}$. Now we obtain for $z \in \mathbb{D}$:

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{a_m(\varphi, \bar{\mathbb{D}})}{(im)^r} z^m \\
&\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{1}{(i\nu)^r} e^{i\nu t} \Bigg) \cdot \Bigg(\sum_{m=1}^{\infty} a_m(\varphi, \bar{\mathbb{D}})\, z^m e^{-imt} \Bigg)\, dt \\
&\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{1}{(i\nu)^r} e^{i\nu t} \Bigg) \cdot \Bigg(\frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{m=1}^{\infty} a_m(\varphi, \bar{\mathbb{D}})\, e^{im(\sigma - t)} \Bigg) \\
&\qquad \times \Bigg(\sum_{\ell=0}^{\infty} z^{\ell} e^{-i\ell\sigma} \Bigg)\, d\sigma \Bigg)\, dt \\
&\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{1}{(i\nu)^r} e^{i\nu t} \Bigg) \cdot \Bigg(\frac{1}{2\pi i} \int_{-\pi}^{\pi} \varphi(e^{i\sigma} e^{-it})\, \frac{d(e^{i\sigma})}{e^{i\sigma} - z} \Bigg)\, dt \\
&\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{1}{(i\nu)^r} e^{i\nu t} \Bigg) \cdot \Bigg(\frac{1}{2\pi i} \int_{|\tau|=1} \frac{\varphi(\tau e^{-it})}{\tau - z}\, d\tau \Bigg)\, dt.
\end{aligned}$$

Thus we have the expansion ($\varphi = F^r f$, $z \in \mathbb{D}$)

$$f(z) = a_0(f, \bar{\mathbb{D}}) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Bigg(\sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{1}{(i\nu)^r} e^{i\nu t} \Bigg) \cdot \Bigg(\frac{1}{2\pi i} \int_{|\tau|=1} \frac{\varphi(\tau e^{-it})}{\tau - z}\, d\tau \Bigg)\, dt.$$

This leads to our next definition (cf. [11, Vol. 1, p. 42]):

DEFINITION 3. Let G be a domain in $\mathbb{C}$ which boundary ∂G can be represented as a rectifiable Jordan curve, $f \in A(\bar{G})$ and $r \in \mathbb{N}$. By φ we denote a function in $A(\bar{G})$ with $\int_{-\pi}^{\pi} \varphi(\psi(\bar{G}, e^{it}))\, dt = 0$.

Then we call φ the rth Faber derivative $F^r f$ of f if it satisfies

$$f(z) = a_0(f, \bar{G}) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\substack{v=-\infty \\ v \neq 0}}^{\infty} \frac{e^{ivt}}{(iv)^r} \right)$$

$$\times \left(\frac{1}{2\pi i} \int_{\partial G} \frac{\varphi(\psi(\bar{G}, \psi^{-1}(\bar{G}, \zeta)\, e^{-it}))}{\zeta - z}\, d\zeta \right) dt \qquad (z \in G).$$

We mention the following Convolution Theorem of Dzjadyk [3, p. 372]:

THEOREM 5. *With notations as in Definition* 3, *if* $\varphi = F^r f$ *then*

$$f(z) = a_0(f, \bar{G}) + \sum_{m=1}^{\infty} \frac{a_m(\varphi, \bar{G})}{(im)^r}\, p_m(\bar{G}, z) \qquad (z \in G), \qquad (*)$$

where the series is pointwise convergent.

Now we give conditions which are sufficient for uniform convergence on $\bar{G}$ of this series.

THEOREM 6. *Let* G, f, r, φ *as in Definition* 3. *Additionally we assume that the boundary curve* ∂G *is of bounded rotation and that the function* $\varphi(\psi(e^{it}))$ *belongs to* $H^\alpha(\mathbb{T})$ *for all* $0 < \alpha \leqslant 1$. *Moreover let* $\varphi = F^r f$. *Then*

(1) *let*

$$\sum_{m=n+1}^{\infty} \frac{a_m(\varphi, \bar{G})\, p_m(\bar{G}, z)}{m^j}$$

$$= \frac{\varphi(z) - s_n(\varphi, \bar{G}, z)}{(n+1)^j} + \left(\frac{1}{(n+1)^j} - \frac{1}{(n+2)^j} \right)$$

$$\times \frac{(n+1)^2}{2n+3} \left(\varphi(z) - R_n^2(\varphi, \bar{G}, z)\right)$$

$$- \sum_{m=n+1}^{\infty} \left(\left(\frac{1}{m^j} - \frac{1}{(m+1)^j} \right) \frac{1}{2m+1} \right.$$

$$\left. - \left(\frac{1}{(m+1)^j} - \frac{1}{(m+2)^j} \right) \frac{1}{2m+3} \right)$$

$$\times (m+1)^2 \left(\varphi(z) - R_m^2(\varphi, \bar{G}, z)\right) \qquad (5.1)$$

for all $n, j \in \mathbb{N}_0$, $z \in \overline{G}$ *and the series on the left side as well as this on the right side converges uniformly on* $\overline{G}$;

(2) *let*

$$\max_{z \in \overline{G}} |f(z) - s_n(f, \overline{G}, z)| \overset{n \to \infty}{=} O\left(\frac{\ln n}{n^{r+\alpha}}\right). \tag{5.2}$$

As an immediate consequence from Dzjadek's Theorem 5 and Theorem 6 we see that $(*)$ holds uniformly in G.

We prove Theorem 6 in Section 8.

6. THE MAIN RESULT

DEFINITION 4. For $r \in \mathbb{N}$ and $\alpha \in]0, 1]$ we define the class $F^r H^\alpha(\overline{G})$ as the set of all functions $f \in A(\overline{G})$ with a Faber derivative $\varphi = F^r f$ of rth order such that $\varphi \circ \psi(e^{it})$ belongs to $H^\alpha(\mathbb{T})$.

In this section we consider matrices of the special form

$$\mu := (\mu_m^n) := \begin{pmatrix} 1 & \mu_1^0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & \mu_1^1 & \mu_2^1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & \mu_1^2 & \mu_2^2 & \mu_3^2 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \mu_1^n & \mu_2^n & \mu_3^n & \mu_4^n & \cdots & \mu_n^n & \mu_{n+1}^n & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Moreover we assume that for $n \in \mathbb{N}_0$ and $0 \leqslant m \leqslant n+1$ we have an expansion

$$\mu_m^n = 1 + \sum_{v=1}^{\infty} b_v(n) \left(\frac{m}{n+1}\right)^v, \tag{6.1}$$

where the coefficients $b_v(n)$ fulfill the condition

$$\Lambda(n) := 1 + \sum_{v=1}^{\infty} v\, |b_v(n)| < \infty \qquad (n \in \mathbb{N}_0). \tag{6.2}$$

Remark. It was discovered by A. K. Pokalo (cf. [3, p. 318]) that the means in the sense of Fejér, Rogosinski and others fulfill the joint conditions (6.1) and (6.2).

By $(*)$ we obtain for the μ-means (as defined in (3)) when $f \in F^r H^\alpha(\overline{G})$ the equation

$$U_n^\mu(f, \overline{G}, z) = a_0(f, \overline{G}) + \sum_{m=1}^{n} \frac{a_m(\varphi, \overline{G})}{(im)^r} \mu_m^n p_m(\overline{G}, z) \qquad (z \in \mathbb{C}). \tag{6.3}$$

Now we are ready for our main result which quantifies the approximation of a function by the μ-means:

THEOREM 7. *Let G be a Jordan domain with rectifiable boundary curve of bounded rotation, $r \in \mathbb{N}$, $\alpha \in]0, 1]$ and $f \in F^r H^\alpha(\bar{G})$. Let the matrix μ be given as above* (*see* (6.1), (6.2)). *Then we have, for all $n \in \mathbb{N}_0$ and $z \in \bar{G}$,*

$$
\begin{aligned}
f(z) - U_n^\mu(f, \bar{G}, z) &= -\sum_{\nu=1}^{r} \frac{b_\nu(n)}{(i(n+1))^\nu} (F^\nu f)(z) + \mu_{n+1}^n \frac{\varphi(z) - s_n(\varphi, \bar{G}, z)}{(i(n+1))^r} \\
&\quad + \left\{ \begin{array}{ll} O\left(\dfrac{|b_{r+1}(n)|}{n^{r+\alpha}}\right) & \text{if} \quad \alpha \in]0, 1[\\ O\left(\dfrac{|b_{r+1}(n)| \ln n}{n^{r+1}}\right) & \text{if} \quad \alpha = 1 \end{array} \right\} \\
&\quad + O\left(\frac{\Lambda(n)}{n^{r+\alpha}}\right) \qquad (n \to \infty).
\end{aligned} \tag{6.4}
$$

Remarks. (1) The condition (6.2) obviously implies the absolute convergence of the series in (6.1).

(2) Theorem 7 also contains full information about the order of saturation in the situation under view. Here we omit a detailed formulation.

We mention some special cases:

(1) For the partial sums $U_n^\mu(f, \bar{G}, z) = s_n(f, \bar{G}, z)$ we obtain for $z \in \bar{G}$

$$f(z) - s_n(f, \bar{G}, z) = \frac{\varphi(z) - s_n(\varphi, \bar{G}, z)}{(i(n+1))^r} + O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n \to \infty).$$

(2) For the Fejér means $U_n^\mu(f, \bar{G}, z) = \sigma_n(f, \bar{G}, z)$ we have for $z \in \bar{G}$

$$f(z) - \sigma_n(f, \bar{G}, z) = \frac{(F^1 f)(z)}{i(n+1)} + O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n \to \infty).$$

(3) For the Riesz means $U_n^\mu(f, \bar{G}, z) = R_n^q(f, \bar{G}, z)$ we have to discuss four cases ($z \in \bar{G}$):

(3.1) $q < r$. Then

$$f(z) - R_n^s(f, \bar{G}, z) = \frac{(F^q f)(z)}{(i(n+1))^q} + O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n \to \infty).$$

(3.2) $q=r$. Then

$$f(z)-R_n^r(f,\bar{G},z)=\frac{\varphi(z)}{(i(n+1))^r}+O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n\to\infty).$$

(3.3.1) $q=r+1$ and $\alpha\in\,]0,1[$. Then

$$f(z)-R_n^{r+1}(f,\bar{G},z)=O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n\to\infty).$$

(3.3.2) $q=r+1$ and $\alpha=1$. Then

$$f(z)-R_n^{r+1}(f,\bar{G},z)=O\left(\frac{\ln n}{n^{r+1}}\right) \qquad (n\to\infty).$$

(3.4) $q>r+1$. Then

$$f(z)-R_n^q(f,\bar{G},z)=O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n\to\infty).$$

(4) For the Rogosinski means $U_n^\mu(f,\bar{G},z)=U_n^{\mathrm{Rog}}(f,\bar{G},z)$ we obtain $(z\in\bar{G})$:

(4.1) In the case of even $r=2j$,

$$f(z)-U_n^{\mathrm{Rog}}(f,\bar{G},z)=-\sum_{k=1}^{j-1}\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}\frac{(F^{2k}f)(z)}{(i(n+1))^{2k}}-\frac{(-1)^j}{r!}\left(\frac{\pi}{2}\right)^r \times\frac{\varphi(z)}{(i(n+1))^r}+O\left(\frac{1}{n^{r+\alpha}}\right) \qquad (n\to\infty).$$

(4.2) In the case of odd $r=2j+1$,

$$f(z)-U_n^{\mathrm{Rog}}(f,\bar{G},z)=-\sum_{k=1}^{j}\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k}\frac{(F^{2k}f)(z)}{(i(n+1))^{2k}} +\begin{cases} O\left(\dfrac{1}{n^{r+\alpha}}\right) & \text{if} \quad 0<\alpha<1,\\ O\left(\dfrac{\ln n}{n^{r+1}}\right) & \text{if} \quad \alpha=1.\end{cases} \tag{6.5}$$

We mention that in the case $r=5$ and $0<\alpha<1$ we obtain (1.1) from (6.5).

7. TECHNICAL PREPARATIONS

In this section we consider some Jordan domain G with rectifiable boundary and some $f \in A(\bar{G})$.

LEMMA 1. *Let* $n, j \in \mathbb{N}_0$, $N \geqslant n+1$, *and* $z \in \bar{G}$. *Then*

$$\sum_{m=n+1}^{N} \frac{a_m(f, \bar{G})\, p_m(\bar{G}, z)}{m^j}$$
$$= \frac{f(z) - s_n(f, \bar{G}, z)}{(n+1)^j} + \left(\frac{1}{(n+1)^j} - \frac{1}{(n+2)^j}\right) \frac{(n+1)^2}{2n+3}$$
$$\times (f(z) - R_n^2(f, \bar{G}, z)) - \sum_{m=n+1}^{N-2} \left[\left(\frac{1}{m^j} - \frac{1}{(m+1)^j}\right) \frac{1}{2m+1}\right.$$
$$\left. - \left(\frac{1}{(m+1)^j} - \frac{1}{(m+2)^j}\right) \frac{1}{2m+3}\right] (m+1)^2 \,(f(z) - R_m^2(f, \bar{G}, z))$$
$$- \frac{f(z) - s_N(f, \bar{G}, z)}{N^j} - \left(\frac{1}{(N-1)^j} - \frac{1}{N^j}\right)$$
$$\times \frac{N^2}{2N-1} (f(z) - R_N^2(f, \bar{G}, z)). \tag{7.1}$$

Proof. The definition of the partial sums s_m gives ($s_{-1} := 0$)

$$a_m(f, \bar{G})\, p_m(\bar{G}, z) = s_m(f, \bar{G}, z) - s_{m-1}(f, \bar{G}, z) \qquad (m \in \mathbb{N}_0) \tag{7.2}$$

and we see

$$\sum_{m=n+1}^{N} \frac{a_m(f, \bar{G})\, p_m(\bar{G}, z)}{m^j}$$
$$= \sum_{m=n+1}^{N} \frac{s_m - s_{m-1}}{m^j}$$
$$= \sum_{m=n+1}^{N} \frac{s_m}{m^j} - \sum_{m=n}^{N-1} \frac{s_m}{(m+1)^j}$$
$$= -\frac{s_n}{(n+1)^j} + \sum_{m=n+1}^{N-1} \left(\frac{1}{m^j} - \frac{1}{(m+1)^j}\right) s_m + \frac{s_N}{N^j}.$$

Using the definition (2.1) of the Riesz means $R_m^2(f, \bar{G}, z)$ we can transform this into (with $R_{-1}^2 := 0$)

$$s_m = \frac{(m+1)^2}{2m+1} R_m^2 - \frac{m^2}{2m+1} R_{m-1}^2 \qquad (m \in \mathbb{N}_0). \tag{7.3}$$

Similar as above this leads to the equation

$$\begin{aligned}
\sum_{m=n+1}^{N} & \frac{a_m(f, \bar{G})\, p_m(\bar{G}, z)}{m^j} \\
&= -\frac{s_n}{(n+1)^j} - \left(\frac{1}{(n+1)^j} - \frac{1}{(n+2)^j}\right) \frac{(n+1)^2}{2n+3} R_n^2 \\
&\quad + \sum_{m=n+1}^{N-2} \left[\left(\frac{1}{m^j} - \frac{1}{(m+1)^j}\right) \frac{1}{2m+1} \right. \\
&\quad \left. - \left(\frac{1}{(m+1)^j} - \frac{1}{(m+2)^j}\right) \frac{1}{2m+3} \right] (m+1)^2 R_m^2 \\
&\quad + \frac{s_N}{N^j} + \left(\frac{1}{(N-1)^j} - \frac{1}{N^j}\right) \frac{N^2}{2N-1} R_N^2.
\end{aligned} \tag{7.4}$$

Now we consider the case $f \equiv 1$ on $\bar{G}$. Then $a_0(1, \bar{G}) = 1$, $a_m(1, \bar{G}) = 0$ for $m \in \mathbb{N}$ and therefore $s_n(1, \bar{G}) = R_n^2(1, \bar{G}) = 1$ for all $n \in \mathbb{N}_0$. By (7.4) we obtain

$$\begin{aligned}
0 &= -\frac{1}{(n+1)^j} - \left(\frac{1}{(n+1)^j} - \frac{1}{(n+2)^j}\right) \frac{(n+1)^2}{2n+3} \\
&\quad + \sum_{m=n+1}^{N-2} \left[\left(\frac{1}{m^j} - \frac{1}{(m+1)^j}\right) \frac{1}{2m+1} \right. \\
&\quad \left. - \left(\frac{1}{(m+1)^j} - \frac{1}{(m+2)^j}\right) \frac{1}{2m+3} \right] (m+1)^2 \\
&\quad + \frac{1}{N^j} + \left(\frac{1}{(N-1)^j} - \frac{1}{N^j}\right) \frac{N^2}{2N-1}.
\end{aligned} \tag{7.5}$$

Now (7.5) and (7.4) gives the desired equation (7.1).

LEMMA 2. *Let $n \in \mathbb{N}_0$, $j \in \mathbb{N}$, and $z \in \bar{G}$. Then*

$$\sum_{m=0}^{n} m^j a_m(f, \bar{G})\, p_m(\bar{G}, z)$$

$$= -n^j(f(z) - s_n(f, \bar{G}, z)) + \frac{n^j - (n-1)^j}{2n-1} n^2(f(z) - R^2_{n-1}(f, \bar{G}, z))$$

$$- \sum_{m=0}^{n-2} \left(\frac{(m+2)^j - (m+1)^j}{2m+3} - \frac{(m+1)^j - m^j}{2m+1} \right)$$

$$\times (m+1)^2\, (f(z) - R^2_m(f, \bar{G}, z)). \tag{7.6}$$

Proof. Lemma 2 can be proved with similar arguments as Lemma 1. Here we have to use (7.2) in the first Abel transform and (7.3) in the second.

8. PROOF OF THEOREM 6

We give the proof of Theorem 6 in three steps.

Step 1. To prove (5.1) we discuss the terms in equality (7.1) (Lemma 1) for $N \to \infty$. From the obvious inequality

$$\frac{1}{m^j} - \frac{1}{(m+1)^j} = -j \int_{m+1}^{m} x^{-j-1}\, dx$$

$$\leqslant -j \int_{m+1}^{m} \frac{1}{m^{j+1}}\, dx = \frac{j}{m^{j+1}} \qquad (j, m \in \mathbb{N})$$

we obtain

$$\left| \frac{1}{m^j} - \frac{1}{(m+1)^j} \right| \leqslant \frac{|j|}{m^{j+1}} \qquad (j \in \{-1\} \cup \mathbb{N}_0,\, m \in \mathbb{N}). \tag{8.1}$$

Similarly we derive

$$\left(\frac{1}{m^j} - \frac{1}{(m+1)^j} \right) \frac{1}{2m+1} - \left(\frac{1}{(m+1)^j} - \frac{1}{(m+2)^j} \right) \frac{1}{2m+3}$$

$$= \int_{m}^{m+1} D\, \frac{(x+1)^{-j} - x^{-j}}{2x+1}\, dx \qquad (j \in \mathbb{N}). \tag{8.2}$$

A simple calculation gives

$$D\,\frac{(x+1)^{-j}-x^{-j}}{2x+1}$$

$$=\frac{(-j-2)((x+1)^{-j}-x^{-j})-jx(x+1)((x+1)^{-j-2}-x^{-j-2})}{(2x+1)^2} \qquad (8.3)$$

and it easily follows that, for $j\in\mathbb{Z}\setminus\{-2,-1,0\}$ (using (8.1))

$$0<D\,\frac{(x+1)^{-j}-x^{-j}}{2x+1}<\frac{j(j+2)}{x^{j+3}}.$$

Therefore we get by (8.2) the inequality

$$0<\left(\frac{1}{m^j}-\frac{1}{(m+1)^j}\right)\frac{1}{2m+1}-\left(\frac{1}{(m+1)^j}-\frac{1}{(m+2)^j}\right)\frac{1}{2m+3}$$

$$<\frac{j(j+2)}{m^{j+3}} \qquad (j\in\mathbb{N}).$$

This, in combination with Theorem 5, shows

$$\max_{z\in\bar{G}}\left|\sum_{m=n+1}^{N-2}\left[\left(\frac{1}{m^j}-\frac{1}{(m+1)^j}\right)\frac{1}{2m+1}-\left(\frac{1}{(m+1)^j}-\frac{1}{(m+2)^j}\right)\frac{1}{2m+3}\right]\right.$$

$$\left.\times(m+1)^2\,(\varphi(z)-R_m^2(\varphi,\bar{G},z))\right|$$

$$\leqslant\sum_{m=n+1}^{N-2}\frac{j(j+2)}{m^{j+3}}\,(m+1)^2\cdot O\left(\frac{1}{m^\alpha}\right)$$

$$=j(j+2)\,O(1)\int_n^{N-2}\frac{dx}{x^{j+1+\alpha}}$$

$$=(j+2)\,O(1)\left(\frac{1}{n^{j+\alpha}}-\frac{1}{(N-2)^{j+\alpha}}\right). \qquad (8.4)$$

Note that the O-symbols are independent of N. So, for $N \to \infty$, we obtain an universal upper bound for the sum

$$\sum_{m=n+1}^{N-2} [\cdots](m+1)^2 (\varphi(z) - R_m^2(\varphi, \bar{G}, z))$$

in (7.1) by

$$\max_{z \in \bar{G}} \sum_{m=n+1}^{\infty} \left[\left(\frac{1}{x^j} - \frac{1}{(x+1)^j} \right) \frac{1}{2x+1} \Bigg|_{m+1}^{m} \right] (m+1)^2 (\varphi(z) - R_m^2(\varphi, \bar{G}, z))$$

$$= (j+2) \cdot O\left(\frac{1}{n^{j+\alpha}}\right) \qquad (n \to \infty). \tag{8.5}$$

Step 2. Next we consider the last terms

$$-\frac{\varphi(z) - s_N(\varphi, \bar{G}, z)}{N^j} - \left(\frac{1}{(N-1)^j} - \frac{1}{N^j}\right) \frac{N^2}{2N-1} (\varphi(z) - R_N^2(\varphi, \bar{G}, z))$$

of Eq. (7.1). By (8.1) and Theorem 5 we have

$$\max_{z \in \bar{G}} \left| \left(\frac{1}{(N-1)^j} - \frac{1}{N^j}\right) \frac{N^2}{2N-1} (\varphi(z) - R_N^2(\varphi, \bar{G}, z)) \right| \overset{N \to \infty}{=} O\left(\frac{1}{N^{j+\alpha}}\right).$$

By Theorem 3 and Theorem 2 we obtain

$$\max_{z \in \bar{G}} \left| \frac{\varphi(z) - s_N(\varphi, \bar{G}, z)}{N^j} \right| \overset{N \to \infty}{=} O\left(\frac{\ln N}{N^{j+\alpha}}\right).$$

Thus from (7.1) we see (5.1) when N tends to ∞.

Step 3. Now we prove (5.2). We have

$$\max_{z \in \bar{G}} |f(z) - s_n(f, \bar{G}, z)| = \max_{z \in \bar{G}} \left| \sum_{m=n+1}^{\infty} \frac{a_m(\varphi, \bar{G})\, p_m(\bar{G}, z)}{(im)^r} \right|.$$

Similar arguments as above, using (5.1), (8.1), (8.5), and Theorem 2.3, and 5, lead to (5.2).

9. PROOF OF THEOREM 7

From (6.3) we see for $z \in \bar{G}$

$$
\begin{aligned}
U_n^\mu(f, \bar{G}, z) &= a_0(f, \bar{G}) + \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{(im)^r} \mu_m^n p_m(\bar{G}, z) \\
&\overset{(6.1)}{=} a_0(f, \bar{G}) + \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{(im)^r} \\
&\quad \times \left(1 + \sum_{\nu=1}^{\infty} b_\nu(n) \left(\frac{m}{n+1}\right)^\nu\right) p_m(\bar{G}, z) \\
&\overset{\text{Remark in Section 6}}{=} a_0(f, \bar{G}) + \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{(im)^r} p_m(\bar{G}, z) \\
&\quad + \sum_{\nu=1}^{\infty} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=1}^{n} \frac{m^\nu a_m(\varphi, \bar{G})}{(im)^r} p_m(\bar{G}, z) \\
&\overset{(*)}{=} f(z) - \sum_{m=n+1}^{\infty} \frac{a_m(\varphi, \bar{G})}{(im)^r} p_m(\bar{G}, z) \\
&\quad + \sum_{\nu=1}^{r-1} \frac{b_\nu(n)}{(i(n+1))^\nu} \left(\sum_{m=1}^{\infty} \frac{a_m(\varphi, \bar{G})}{(im)^{r-\nu}} p_m(\bar{G}, z) \right. \\
&\quad \left. - \sum_{m=n+1}^{\infty} \frac{a_m(\varphi, \bar{G})}{(im)^{r-\nu}} p_m(\bar{G}, z) \right) \\
&\quad + i^{-r} \sum_{\nu=r}^{\infty} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{m^{r-\nu}} p_m(\bar{G}, z) \\
&\overset{\text{Def. 3, Thm. 5}}{=} f(z) + \sum_{\nu=1}^{r-1} \frac{b_\nu(n)}{(i(n+1))^\nu} (F^\nu f)(z) \\
&\quad - i^{-r} \sum_{\nu=0}^{r-1} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=n+1}^{\infty} \frac{a_m(\varphi, \bar{G})}{m^{r-\nu}} p_m(\bar{G}, z) \\
&\quad + i^{-r} \sum_{\nu=r}^{\infty} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{m^{r-\nu}} p_m(\bar{G}, z).
\end{aligned}
$$

Therefore, using the uniform convergence of the series, we get

$$f(z) - U_n^\mu(f, \bar{G}, z) = -\sum_{\nu=1}^{r-1} \frac{b_\nu(n)}{(i(n+1))^\nu} (F^\nu f)(z)$$
$$+ i^{-r} \sum_{\nu=0}^{r-1} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=n+1}^{\infty} \frac{a_m(\varphi, \bar{G})}{m^{r-\nu}} p_m(\bar{G}, z)$$
$$- i^{-r} \sum_{\nu=r}^{\infty} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{m^{r-\nu}} p_m(\bar{G}, z). \tag{9.1}$$

Now we discuss the three sums of the right side step by step.

Step 1. From Lemma 2 we obtain

$$-i^{-r} \sum_{\nu=r}^{\infty} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=1}^{n} \frac{a_m(\varphi, \bar{G})}{m^{r-\nu}} p_m(\bar{G}, z)$$
$$= -\frac{b_r(n)}{(i(n+1))^r} (s_n(\varphi, \bar{G}, z) - \varphi(z))$$
$$- i^{-r} \frac{b_{r+1}(n)}{(n+1)^{r+1}} \Bigg(-n(\varphi(z) - s_n(\varphi, \bar{G}, z))$$
$$+ \frac{n^2}{2n-1} (\varphi(z) - R_{n-1}^2(\varphi, \bar{G}, z))$$
$$+ 2 \sum_{m=0}^{n-2} \frac{(m+1)^2}{(2m+1)(2m+3)} (\varphi(z) - R_m^2(\varphi, \bar{G}, z)) \Bigg)$$
$$- i^{-r} \frac{b_{r+2}(n)}{(n+1)^{r+2}} \Bigg(-n^2(\varphi(z) - s_n(\varphi, \bar{G}, z))$$
$$+ n^2(\varphi(z) - R_{n-1}^2(\varphi, \bar{G}, z)) \Bigg)$$
$$- i^{-r} \sum_{\nu=r+3}^{\infty} \frac{b_\nu(n)}{(n+1)^\nu} \Bigg(-n^{\nu-r}(\varphi(z) - s_n(\varphi, \bar{G}, z))$$
$$+ \frac{n^{\nu-r} - (n-1)^{\nu-r}}{2n-1} n^2(\varphi(z) - R_{n-1}^2(\varphi, \bar{G}, z))$$
$$- \sum_{m=0}^{n-2} \left(\frac{(m+2)^{\nu-r} - (m+1)^{\nu-r}}{2m+3} - \frac{(m+1)^{\nu-r} - m^{\nu-r}}{2m+1} \right)$$
$$\times (m+1)^2 (\varphi(z) - R_m^2(\varphi, \bar{G}, z)) \Bigg).$$

Step 2.

$$i^{-r}\sum_{\nu=0}^{r-1}\frac{b_\nu(n)}{(n+1)^\nu}\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(n+1)^{r-\nu}}-\frac{b_r(n)}{(i(n+1))^r}s_n(\varphi,\bar{G},z)$$

$$-i^{-r}\sum_{\nu=r+1}^{\infty}\frac{b_\nu(n)}{(n+1)^\nu}(-n^{\nu-r}(\varphi(z)-s_n(\varphi,\bar{G},z)))$$

$$=\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(i(n+1))^r}\sum_{\nu=0}^{r-1}b_\nu(n)+\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(i(n+1))^r}b_r(n)$$

$$-\frac{b_r(n)}{(i(n+1))^r}\varphi(z)+i^{-r}(\varphi(z)-s_n(\varphi,\bar{G},z))$$

$$\times\sum_{\nu=r+1}^{\infty}\frac{(n+1)^{\nu-r}-((n+1)^{\nu-r}-n^{\nu-r})}{(n+1)^\nu}b_\nu(n)$$

$$=-\frac{b_r(n)}{(i(n+1))^r}\varphi(z)+\mu_{n+1}^n\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(i(n+1))^r}$$

$$-\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(i(n+1))^r}\sum_{\nu=r+1}^{\infty}\left(1-\left(\frac{n}{n+1}\right)^{\nu-r}\right)b_\nu(n).$$

Step 3. Now we begin the proof of (6.4). By (9.1) we obtain

$$f(z)-U_n^\mu(f,\bar{G},z)$$

$$=-\sum_{\nu=1}^{r-1}\frac{b_\nu(n)}{(i(n+1))^\nu}(F^\nu f)(z)-\frac{b_r(n)}{(i(n+1))^r}\varphi(z)$$

$$+\mu_{n+1}^n\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(i(n+1))^r}$$

$$-2i^{-r}\frac{b_{r+1}(n)}{(n+1)^r}\sum_{m=0}^{n-2}\frac{(m+1)^2}{(2m+1)(2m+3)}(\varphi(z)-R_m^2(\varphi,\bar{G},z))$$

$$-\frac{\varphi(z)-s_n(\varphi,\bar{G},z)}{(i(n+1))^r}\sum_{\nu=r+1}^{\infty}\left(1-\left(\frac{n}{n+1}\right)^{\nu-r}\right)b_\nu(n)$$

$$+i^{-r}\sum_{\nu=0}^{r-1}\frac{b_\nu(n)}{(n+1)^\nu}\left(\frac{1}{(n+1)^{r-\nu}}-\frac{1}{(n+2)^{r-\nu}}\right)$$

$$\times\frac{(n+1)^2}{2n+3}(\varphi(z)-R_n^2(\varphi,\bar{G},z))$$

$$
-i^{-r}\sum_{\nu=0}^{r-1}\frac{b_\nu(n)}{(n+1)^\nu}\sum_{m=n+1}^{\infty}\left[\left(\frac{1}{x^{r-\nu}}-\frac{1}{(x+1)^{r-\nu}}\right)\frac{1}{2x+1}\Bigg|_{m+1}^{m}\right]
$$
$$
\times(m+1)^2\,(\varphi(z)-R_m^2(\varphi,\bar{G},z))
$$
$$
-i^{-r}\sum_{\nu=r+1}^{\infty}\frac{b_\nu(n)}{(n+1)^\nu}\,\frac{n^{\nu-r}-(n-1)^{\nu-r}}{2n-1}\,n^2(\varphi(z)-R_{n-1}^2(\varphi,\bar{G},z))
$$
$$
+i^{-r}\sum_{\nu=r+3}^{\infty}\frac{b_\nu(n)}{(n+1)^\nu}\sum_{m=0}^{n-2}\left(\frac{(m+2)^{\nu-r}-(m+1)^{\nu-r}}{2m+3}\right.
$$
$$
\left.-\frac{(m+1)^{\nu-r}-m^{\nu-r}}{2m+1}\right)(m+1)^2\,(\varphi(z)-R_m^2(\varphi,\bar{G},z)). \tag{9.2}
$$

Step 4. We discuss the cases depending on α.

Case $\alpha\in\,]0,1[$. Then we have

$$
\sum_{m=0}^{n-2}\frac{1}{(m+2)^\alpha}<\int_0^n\frac{dx}{x^\alpha}=\frac{n^{1-\alpha}}{1-\alpha}.
$$

Case $\alpha=1$. Then

$$
\sum_{m=0}^{n-2}\frac{1}{m+2}<\int_1^n\frac{dx}{x}=\ln n.
$$

Thus, using Theorem 5, we can estimate the term in Step 3 in both cases by

$$
\max_{z\in\bar{G}}\left|2i^{-r}\frac{b_{r+1}(n)}{(n+1)^r}\sum_{m=0}^{n-2}\frac{(m+1)^2}{(2m+1)(2m+3)}(\varphi(z)-R_m^2(\varphi,\bar{G},z))\right|
$$
$$
=\frac{|b_{r+1}(n)|}{(n+1)^{r+1}}\sum_{m=0}^{n-2}O\left(\frac{1}{(m+2)^\alpha}\right)
$$
$$
=|b_{r+1}(n)|\begin{cases}O\left(\dfrac{1}{n^{r+\alpha}}\right) & \text{if}\quad \alpha\in\,]0,1[\\[2ex] O\left(\dfrac{\ln n}{n^{r+1}}\right) & \text{if}\quad \alpha=1\end{cases}
$$

Step 5. From Bernoulli's inequality we see

$$
1-\left(\frac{n}{n+1}\right)^j\leqslant 1-\left(1-\frac{j}{n+1}\right)=\frac{j}{n+1}\qquad (j\in\mathbb{N},\,n\in\mathbb{N}_0).
$$

Now, using Theorems 3 and 2 we obtain

$$\max_{z \in \bar{G}} \left| \frac{\varphi(z) - s_n(\varphi, \bar{G}, z)}{(i(n+1))^r} \sum_{\nu=r+1}^{\infty} \left(1 - \left(\frac{n}{n+1}\right)^{\nu-r}\right) b_\nu(n) \right|$$

$$= O\left(\frac{\ln n}{n^{r+\alpha}}\right) \frac{1}{n+1} \sum_{\nu=r+1}^{\infty} |\nu - r| \, |b_\nu(n)|$$

$$= O\left(\frac{1}{n^{r+\alpha}}\right) \frac{\ln(n+2)}{n+1} \sum_{\nu=r+1}^{\infty} |\nu - r| \, |b_\nu(n)| = o\left(\frac{\Lambda(n)}{n^{r+\alpha}}\right).$$

Step 6. Inequality (8.1) and Theorem 5 show

$$\max_{z \in \bar{G}} \left| i^{-r} \sum_{\nu=0}^{r-1} \frac{b_\nu(n)}{(n+1)^\nu} \left(\frac{1}{(n+1)^{r-\nu}} - \frac{1}{(n+2)^{r-\nu}}\right) \right.$$

$$\left. \times \frac{(n+1)^2}{2n+3} (\varphi(z) - R_n^2(\varphi, \bar{G}, z)) \right|$$

$$= \sum_{\nu=0}^{r-1} \frac{b_\nu(n)}{(n+1)^\nu} (r-\nu) \, O\left(\frac{1}{n^{r-\nu+\alpha}}\right) = O\left(\frac{1}{n^{r+\alpha}}\right) \sum_{\nu=0}^{r-1} |\nu - r| \, |b_\nu(n)|.$$

Step 7. The inequality (8.5) leads to

$$\max_{z \in \bar{G}} \left| i^{-r} \sum_{\nu=0}^{r-1} \frac{b_\nu(n)}{(n+1)^\nu} \sum_{m=n+1}^{\infty} \left[\left(\frac{1}{x^{r-\nu}} - \frac{1}{(x+1)^{r-\nu}}\right) \frac{1}{2x+1} \Big|_{m+1}^{m} \right] \right.$$

$$\left. \times (m+1)^2 (\varphi(z) - R_m^2(\varphi, \bar{G}, z)) \right|$$

$$= O\left(\frac{1}{n^{r+\alpha}}\right) \sum_{\nu=0}^{r-1} |\nu - r - 2| \, |b_\nu(n)|.$$

Step 8. The following inequality is easy to verify

$$(m+1)^j - m^j = j \int_m^{m+1} x^{j-1} \, dx \leqslant j \int_m^{m+1} (m+1)^{j-1} \, dx$$

$$= j(m+1)^{j-1} \qquad (j \in \mathbb{N}, m \in \mathbb{N}_0). \tag{9.3}$$

Thus from Theorem 5 we see

$$\begin{aligned}\max_{z\in\bar{G}}\left|i^{-r}\sum_{\nu=r+1}^{\infty}\frac{b_\nu(n)}{(n+1)^\nu}\frac{n^{\nu-r}-(n-1)^{\nu-r}}{2n-1}n^2(\varphi(z)-R_{n-1}^2(\varphi,\bar{G},z))\right|\\ =\sum_{\nu=r+1}^{\infty}\frac{|b_\nu(n)|}{(n+1)^\nu}\frac{(\nu-r)\,n^{\nu-r-1}}{2n-1}n^2 O\left(\frac{1}{n^\alpha}\right)\\ =O\left(\frac{1}{n^{r+\alpha}}\right)\sum_{\nu=r+1}^{\infty}|\nu-r|\,|b_\nu(n)|.\end{aligned}$$

Step 9. On the analogy to our considerations in the first step of the proof of Theorem 6 we have

$$\begin{aligned}0<\frac{(m+2)^j-(m+1)^j}{2m+3}-\frac{(m+1)^j-m^j}{2m+1}\\ <\frac{2j(j-2)(m+2)^{j-1}}{(2m+1)^2}\qquad (j\geqslant 3).\end{aligned}\tag{9.4}$$

Now let $\nu-r\geqslant 3$ and $\alpha\in\,]0,1]$ be given. Then

$$\sum_{m=0}^{n-2}(m+2)^{\nu-r-1-\alpha}<\int_0^n x^{\nu-r-1-\alpha}\,dx=\frac{n^{\nu-r-\alpha}}{\nu-r-\alpha}.$$

Thus by (9.4) and Theorem 5 we can estimate

$$\begin{aligned}\max_{z\in\bar{G}}\left|i^{-r}\sum_{\nu=r+3}^{\infty}\frac{b_\nu(n)}{(n+1)^\nu}\sum_{m=0}^{n-2}\left(\frac{(x+1)^{\nu-r}-x^{\nu-r}}{2x+1}\Big|_m^{m+1}\right)\right.\\ \left.\times(m+1)^2\,(\varphi(z)-R_m^2(\varphi,\bar{G},z))\right|\\ =\sum_{\nu=r+3}^{\infty}\frac{b_\nu(n)}{(n+1)^\nu}\sum_{m=0}^{n-2}\frac{2(\nu-r)(\nu-r-2)(m+2)^{\nu-r-1}}{(2m+1)^2}\\ \times(m+1)^2\,O\left(\frac{1}{(m+2)^\alpha}\right)\\ =O(1)\sum_{\nu=r+3}^{\infty}\frac{(\nu-r)(\nu-r-2)\,|b_\nu(n)|}{(n+1)^\nu}\sum_{m=0}^{n-2}(m+2)^{\nu-r-1-\alpha}\\ =O(1)\sum_{\nu=r+3}^{\infty}\frac{(\nu-r)(\nu-r-2)\,|b_\nu(n)|}{(\nu-r-\alpha)(n+1)^\nu}n^{\nu-r-\alpha}\\ =O\left(\frac{1}{n^{r+\alpha}}\right)\sum_{\nu=r+3}^{\infty}|\nu-r|\,|b_\nu(n)|.\end{aligned}$$

From (9.2) and Steps 4 to 9 we see the assertion of Theorem 7.

10. FINAL REMARKS

(1) The Zygmund class $Z(\mathbb{T})$ is defined as the set of all continuous functions $g\colon \mathbb{T} \to \mathbb{C}$ with the property

$$\left| g(x) - 2g\left(\frac{x+y}{2}\right) + g(y) \right| \leqslant \text{const} \cdot |x-y| \qquad (x, y \in \mathbb{T}).$$

For $r \in \mathbb{N}$ let $F^r Z(\bar{G})$ denote the class of functions $f \in A(\bar{G})$ with a Faber derivative $F^r f$ of rth order such that $(F^r f) \circ \psi \circ \exp \circ i \in Z(\mathbb{T})$ (compare Definition 4). For this class the result (6.4) of Theorem 7 holds in the form

$$\begin{aligned} f(z) - U_n^{\mu}(f, \bar{G}, z) = &-\sum_{\nu=1}^{r-1} \frac{b_\nu(n)}{(i(n+1))^\nu} (F^\nu f)(z) - \frac{b_r(n)}{(i(n+1))^r} \varphi(z) \\ &+ \mu_{n+1}^n \frac{\varphi(z) - s_n(\varphi, \bar{G}, z)}{(i(n+1))^r} \\ &+ O\left(\frac{|b_{r+1}(n)| \ln n}{n^{r+1}}\right) + O\left(\frac{\Lambda(n)}{n^{r+1}}\right) \qquad (n \to \infty). \end{aligned}$$

(2) Equations of the form (1.1) were first studied in 1932 by E. V. Voronovskaja in a paper about approximation by Bernstein polynomials. Since then her results had been extended in several directions by many authors. We only mention articles of S. N. Bernstein (1932), I. P. Natanson (1944), P. P. Korovkin (1953), R. Taberski (1958), I. M. Petrov (1958), R. G. Mamedov (1959), Y. Matsuoka (1960), P. L. Butzer and E. Görlich (1966). More details can be found in [2]. For recent results on these questions we refer the reader to the monography of V. K. Dzjadyk [3]. A very recent paper is due to V. A. Baskakov [1].

(3) The case of domains with $\mathscr{C}^{1,\alpha}$-boundary (cf. [7, p. 49]) was treated by Bruj in 1974 (cf. [3, p. 383]). It is known that the class of these domains is not contained in the class of all domains which boundary curve is of bounded rotation nor vice versa.

(4) Finally we mention that the considerations in Section 3 above presents new (and simpler) proofs for former results (cf. [2, 3]).

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